GENERAL SHIFTS in MATHEMATICS

1. **FOCUS: In each grade or course, focus on 2-4 topics.**

2. **COHERENCE: Think across grades and link major topics within grades.**

3. **RIGOR: In major topics pursue with equal intensity**
   - **Conceptual Understanding**
   - **Procedural Skill**
   - **Applications**

Rigor = *Conceptual understanding + Procedural skill and fluency + Application*

Shift #3: Rigor requires a balance of the three discrete components of math instruction: conceptual understanding, procedural skills and fluency, and application. This is not simply a pedagogical option, but is required by the Standards. The majority of the Standards specifically call for conceptual understanding, fluency, or application, but not every standard will necessarily fit neatly into just one of these three discrete components. For example, certain standards can be said to require procedural skill and conceptual understanding.

Grade	Standard	Procedural skill and Conceptual Understanding Standards Examples
3	3.G.2	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*
6	6.EE.1	Write and evaluate numerical expressions involving whole-number exponents. *For example, multiply by powers of 10 and products of numbers using exponents* $(7 \bullet 7 \bullet 7 = 7^3)$.

Conceptual understanding: The Standards call for conceptual understanding of key concepts, such as place value and ratios. Teachers support students' ability to access concepts from a number of perspectives so that students are able to see math as more than a set of mnemonics or discrete procedures. Conceptual understanding standards often use the terms "understand" and "recognize."

Grade	Standard	Deep Conceptual Understanding Standards Examples
3	3.NBT.1	Use place value understanding to round whole numbers to the nearest 10 or 100.
6	6.NS.5	Understand that positive and negative numbers describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explain the meaning of 0 in each situation.

Procedural skill and fluency: The Standards call for speed and accuracy in calculation. Teachers structure class time and/or homework time for students to practice core functions such as single-digit multiplication so that students have access to more complex concepts and procedures. Fluency standards clearly state “fluently” in the standard.

Grade	Required Fluency	Standard
K	Add/subtract up to 5	K.OA.5
1	Add/subtract up to 10	1.OA.6
2	Add/subtract up to 20 (know single-digit sums from memory) Add/subtract up to 100	2.OA.2 2.NBT.5
3	Multiply/divide up to 100 (know single-digit products from memory) Add/subtract up to 1000	3.OA.7 3.NBT.2
4	Add/subtract up to 1,000,000	4.NBT.4
5	Multi-digit multiplication	5.NBT.5
6	Multi-digit division Multi-digit decimal operations	6.NS.2,3

Application: The Standards call for students to use math flexibly for applications. Teachers provide opportunities for students to apply math in context. Teachers in content areas outside of math, particularly science, ensure that students are using math to access and make meaning of content. Application standards typically state “apply” or “solve.”

Grade	Standard	Application Grade 3 and 6 Standards Examples
3	3.MD.1	Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes or hours (e.g., by representing the problem on a number line diagram or clock).
6	6.SP.4	Display numerical data in plots on a number line, including dot or line plots, histograms, and box (box and whisker) plots.

Standards for Mathematical Practice

Specific expectations for grade bands K-2, 3-5, 6-8 and 9-12 can be found starting on page 97 of the Alaska English/Language Arts and Mathematics Standards document.

	Mathematically proficient students will:
1. Make sense of problems and persevere in solving them.	• explain the meaning of the problem to themselves. • look for a way to start and note the strategies that will help solve the problem. • identify and analyze givens, constraints, relationships and goals. • make inferences about the form and meaning of the solution. • design a plan to solve the problem. • use effective problem solving strategies. • evaluate the progress and change the strategy if necessary. • solve the problem using a different methods and compare solutions. • ask, "Does this make sense?"

2. Reason abstractly and quantitatively.	• make sense of quantities and their relationships in problem solutions.
	• use two complementary abilities when solving problems involving number relationships.
	    o Decontextualize- be able to reason abstractly and represent a situation symbolically and manipulate the symbols
	    o Contextualize- make meaning of the symbols in the problem
	• understand the meaning of quantities and are flexible in the use of operations and their properties.
	• create a logical representation of the problem.
	• attends to the meaning of quantities, not just how to compute them.

3. Construct viable arguments and critique the reasoning of others.	• analyze problems and use stated mathematical assumptions, definitions, and established results in construction arguments. • justify conclusions with mathematical ideas. • listen to arguments of others and ask useful question to determine if an argument makes sense. • ask clarifying questions or suggest ideas to improve/revise the argument. • compare two arguments and determine correct or flawed logic.

4. Model with mathematics.	• understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize). • apply the math they know to solve problems in everyday life. • are able to simplify a complex problem and identify important quantities to look at relationships. • represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation. • reflect on whether the results make sense possibly improving/revising the model. • ask, "How can I represent this mathematically?"
5. Use appropriate tools strategically.	• use available tools recognizing the strengths and limitations of each. • use estimation and other mathematical knowledge to detect possible errors. • identify relevant external mathematical resources to pose and solve problems. • use technological tools to deepen their understanding of mathematics.
6. Attend to precision.	• communicate precisely with others and try to use clear mathematical language when discussing their reasoning. • understand meanings of symbols used in mathematics and can label quantities appropriately. • express numerical answers with a degree of precision appropriate for the problem context. • calculate efficiently and accurately.
7. Look for and make use of structure.	• apply general mathematical rules to specific situations. • look for overall structure and patterns in mathematics. • see complicated things as a single object or as being composed of several objects. • be able to look at problems from a different perspective.
8. Look for and express regularity in repeated reasoning	• see repeated calculations and look for generalizations and shortcuts. • see the overall process of the problem and still attend to details. • understand the broader application of patterns and see the structure in similar situations. • continually evaluate the reasonableness of their intermediate results.

#1 Make sense of problems and persevere in solving them.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
1. Make sense of problems and persevere in solving them. • Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem. • Plan a solution pathway instead of jumping to a solution. • Monitor the progress and change the approach if necessary. • See relationships between various representations. • Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another. • Students ask themselves, "Does this make sense?" and understand various approaches to solutions.	How would you describe the problem in your own words? How would you describe what you are trying to find? What do you notice about...? What information is given in the problem? Describe the relationship between the quantities. Describe what you have already tried. What might you change? Talk me through the steps you've used to this point. What steps in the process are you most confident about? What are some other strategies you might try? What are some other problems that are similar to this one? How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- ☐ Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available. The problem focuses students' attention on a mathematical idea and provides an opportunity to develop and/or use mathematical habits of mind.
- ☐ Allows for multiple entry points and solution paths as well as, multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- ☐ Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- ☐ Requires students to defend and justify their solutions.

Teacher:

- ☐ Allows students time to initiate a plan; uses question prompts as needed to assist students in developing a pathway.
- ☐ Continually asks students if their plans and solutions make sense.
- ☐ Questions students to see connections to previous solution attempts and/or tasks to make sense of current problem.
- ☐ Consistently asks to defend and justify their solution by comparing solution paths.
- ☐ Differentiates to keep advanced students challenged during work time

#2 Reason abstractly and quantitatively.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
2. Reason abstractly and quantitatively. • Make sense of quantities and their relationships. • Able to decontextualize (represent a situation symbolically and manipulate the symbols) and contextualize (make meaning of the symbols in a problem) quantitative relationships. • Understand the meaning of quantities and are flexible in the use of operations and their properties. • Create a logical representation of the problem. • Attend to the meaning of quantities, not just how to compute them.	What do the numbers used in the problem represent? What is the relationship of the quantities? How is _______ related to ________? What is the relationship between ______and ______? What does_______mean to you? (e.g. symbol, quantity, diagram) What properties might we use to find a solution? How did you decide in this task that you needed to use...? Could you have used another operation or property to solve this task? Why or why not?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Includes questions that require students to attend to the meaning of quantities and their relationships, not just how to compute them.
- ☐ Consistently expects students to convert situations into symbols in order to solve the problem and then requires students to explain the solution within a meaningful situation.
- ☐ Contains relevant, realistic content.

Teacher:

- ☐ Expects students to interpret, model, and connect multiple representations.
- ☐ Asks students to explain the meaning of the symbols in the problem and in their solution.
- ☐ Expects students to give meaning to all quantities in the task.
- ☐ Questions students so that understanding of the relationships between the quantities and/or the symbols in the problem and the solution are fully understood.

#3 Construct viable arguments and critique the reasoning of others.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
3. Construct viable arguments and critique the reasoning of others. • Analyze problems and use stated mathematical assumptions, definitions, and established results in constructing arguments. • Justify conclusions with mathematical ideas. • Listen to the arguments of others and ask useful questions to determine if an argument makes sense. • Ask clarifying questions or suggest ideas to improve/revise the argument. • Compare two arguments and determine correct or flawed logic.	What mathematical evidence supports your solution? How can you be sure that...? / How could you prove that...? Will it still work if...? What were you considering when...? How did you decide to try that strategy? How did you test whether your approach worked? How did you decide what the problem was asking you to find? (What was unknown?) Did you try a method that did not work? Why didn't it work? Would it ever work? Why or why not? What is the same and what is different about...? How could you demonstrate a counter-example?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Is structured to bring out multiple representations, approaches, or error analysis.
- ☐ Embeds discussion and communication of reasoning and justification with others.
- ☐ Requires students to provide evidence to explain their thinking beyond merely using computational skills to find a solution.
- ☐ Expects students to give feedback and ask questions of others' solutions.

Teacher:

- ☐ Encourages students to use proven mathematical understandings, (definitions, properties, conventions, theorems, etc.), to support their reasoning.
- ☐ Questions students so they can tell the difference between assumptions and logical conjectures.
- ☐ Asks questions that require students to justify their solution and their solution pathway.
- ☐ Prompts students to respectfully evaluate peer arguments when solutions are shared.
- ☐ Asks students to compare and contrast various solution methods.
- ☐ Creates various instructional opportunities for students to engage in mathematical discussions (whole group, small group, partners, etc.).

#4 Model with mathematics.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
4. Model with mathematics. • Understand this is a way to reason quantitatively and abstractly (able to decontextualize and contextualize). • Apply the math students know to solve problems in everyday life. • Able to simplify a complex problem and identify important quantities to look at relationships. • Represent mathematics to describe a situation either with an equation or a diagram and interpret the results of a mathematical situation. • Reflect on whether the results make sense, possibly improving/revising the model. • Ask themselves, "How can I represent this mathematically?"	What number model could you construct to represent the problem? What are some ways to represent the quantities? What's an equation or expression that matches the diagram? number line? chart? table? Where did you see one of the quantities in the task in your equation or expression? Would it help to create a diagram, graph, table, …? What are some ways to visually represent…? What formula might apply in this situation?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Is structured so that students represent the problem situation and their solution symbolically, graphically, and/or pictorially (may include technological tools) appropriate to the context of the problem.
- ☐ Invites students to create a context (real-world situation) that explains numerical/symbolic representations.
- ☐ Asks students to take complex mathematics and make it simpler by creating a model that will represent the relationship between the quantities.
- ☐ Requires students to identify variables, compute and interpret results, report findings, and justify the reasonableness of their results and procedures within context of the task.

Teacher:

- ☐ Demonstrates and provides student's experiences with the use of various mathematical models.
- ☐ Questions students to justify their choice of model and the thinking behind the model.
- ☐ Asks students about the appropriateness of the model chosen.
- ☐ Assists students in seeing and making connections among models.
- ☐ Give students opportunity to evaluate the appropriateness of the model.

#5 Use appropriate tools strategically.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
5. Use appropriate tools strategically. • Use available tools recognizing the strengths and limitations of each. • Use estimation and other mathematical knowledge to detect possible errors. • Identify relevant external mathematical resources to pose and solve problems. • Use technological tools to deepen their understanding of mathematics. • Use mathematical models for visualizing and analyzing information	What mathematical tools could we use to visualize and represent the situation? What information do you have? What do you know that is not stated in the problem? What approach are you considering trying first? What estimate did you make for the solution? In this situation would it be helpful to use.a graph..., number line..., ruler..., diagram..., calculator..., manipulative? Why was it helpful to use ____? What can using a ______ show us that _____may not? In what situations might it be more informative or helpful to use...?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Requires multiple learning tools. (Tools may include: manipulatives (concrete models), calculator, measurement tools, graphs, diagrams, spreadsheets, statistical software, etc.)
- ☐ Requires students to determine and use appropriate tools to solve problems.
- ☐ Requires students to demonstrate fluency in mental computations.
- ☐ Asks students to estimate in a variety of situations:
  - -a task when there is no need to have an exact answer
  - -a task when there is not enough information to get an exact answer
  - -a task to check if the answer from a calculation is reasonable

Teacher:

- ☐ Demonstrates and provides students experiences with the use of various math tools. A variety of tools are within the classroom learning environment and readily available.
- ☐ Allows students to choose appropriate learning tools and questions students as to why they chose the tools they used to solve the problem.
- ☐ Consistently models how and when to estimate effectively, and requires students to use estimation strategies in a variety of situations.
- ☐ Asks student to explain their mathematical thinking with the chosen tool.
- ☐ Asks students to explore other options when some tools are not available.

#6 Attend to precision.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
6. Attend to precision. • Communicate precisely with others and try to use clear mathematical language when discussing their reasoning. • Understand meanings of symbols used in mathematics and can label quantities appropriately. • Express numerical answers with a degree of precision appropriate for the problem context. • Calculate efficiently and accurately.	What mathematical terms apply in this situation? How did you know your solution was reasonable? Explain how you might show that your solution answers the problem. Is there a more efficient strategy? How are you showing the meaning of the quantities? What symbols or mathematical notations are important in this problem? What mathematical language..., definitions..., properties can you use to explain...? How could you test your solution to see if it answers the problem?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Requires students to use precise vocabulary (in written and verbal responses) when communicating mathematical ideas.
- ☐ Expects students to use symbols appropriately.
- ☐ Embeds expectations of how precise the solution needs to be (some may more appropriately be estimates).

Teacher:

- ☐ Consistently demands and models precision in communication and in mathematical solutions. (*uses and models correct content terminology*).
- ☐ Expects students to use precise mathematical vocabulary during mathematical conversations. (*identifies incomplete responses and asks students to revise their response*).
- ☐ Questions students to identify symbols, quantities, and units in a clear manner

#7 Look for and make use of structure.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
7. Look for and make use of structure. • Apply general mathematical rules to specific situations. • Look for the overall structure and patterns in mathematics. • See complicated things as single objects or as being composed of several objects.	What observations do you make about...? What do you notice when...? What parts of the problem might you eliminate? simplify? What patterns do you find in...? How do you know if something is a pattern? What ideas have we learned before that were useful in solving this problem? What are some other problems that are similar to this one? How does this relate to...? In what ways does this problem connect to other mathematical concepts?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

<u>Task:</u>

☐ Requires students to look for the structure within mathematics in order to solve the problem. (i.e. – decomposing numbers by place value, working with properties, etc.).

☐ Asks students to take a complex idea and then identify and use the component parts to solve problems. i.e. building on the structure of equal sharing, students connect their understanding to the traditional division algorithm. When “unit size” cannot be equally distributed, it is necessary to break down into a smaller “unit size”. (example below)

$$\begin{array}{r} 4\,\overline{)351} \\ \underline{-32} \\ 31 \\ \underline{-28} \\ 3 \end{array}$$

3 *hundreds* units cannot be distributed into 4 equal groups. Therefore, they must be broken down into *tens* units.

There are now 35 *tens* units to distribute into 4 groups. Each group gets 8 sets of tens, leaving 3 extra *tens* units that need to become *ones* units.

This leaves 31 *ones* units to distribute into 4 groups. Each group gets 7 *ones* units, with 3 *ones* units remaining. The quotient means that each group has 87 with 3 left.

☐ Expects students to look at problems and think about them in an unconventional way that demonstrates a deeper understanding of the mathematical structure—leading to a more efficient way of solving the problem. They recognize and identify structures from previous experience(s) and apply this understanding in a new situation. (i.e. 7 x 8 = (7 x 5) + (7 x 3) OR 7 x 8 = (7 x 4) + (7 x 4). New situations could be distributive property, area of composite figures, multiplication fact strategies.)

<u>Teacher:</u>

☐ Encourages students to look at or something they recognize and have students apply the information in identifying solution paths (i.e. composing/decomposing numbers and geometric figures, identifying properties, operations, etc.).

☐ Expects students to explain the overall structure of the problem and the *big math idea* used to solve the problem.

Secondary Example end of document

#8 Look for and express regularity in repeated reasoning.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
8. Look for and express regularity in repeated reasoning. • See repeated calculations and look for generalizations and shortcuts. • See the overall process of the problem and still attend to the details. • Understand the broader application of patterns and see the structure in similar situations. • Continually evaluate the reasonableness of their intermediate results.	Will the same strategy work in other situations? Is this always true, sometimes true or never true? How would you prove that...? What do you notice about...? What is happening in this situation? What would happen if...? Is there a mathematical rule for...? What predictions or generalizations can this pattern support? What mathematical consistencies do you notice?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Addresses and connects to prior knowledge in a non-routine way.
- ☐ Present several opportunities to reveal patterns or repetition in thinking so generalizations can be made.
- ☐ Requires students to see patterns or relationships in order to develop a mathematical rule.
- ☐ Expects students to discover the underlying structure of the problem and come to a generalization.
- ☐ Connects to a previous task to extend learning of a mathematical concept.

Teacher:

- ☐ Encourages students to connect task to prior concepts and tasks.
- ☐ Prompts students to generate exploratory questions based on current tasks.
- ☐ Asks what math relationships or patterns can be used to assist in making sense of the problem.
- ☐ Asks for predictions about solutions at midpoints throughout the solution process and encourages students to monitor each other's intermediate results.
- ☐ Questions students to assist them in creating generalizations based on repetition in thinking and procedures.

 # 7 Look for and make use of structure

Secondary Examples:

What does it mean to look for and make use of structure?

- Students can look at problems and think about them in an unconventional way that demonstrates a deeper understanding of the mathematical structure – leading to a more efficient means to solving the problem.

Example problem:

- Solve for x : $3(x - 2) = 9$

Rather than approach the problem above by distributing or dividing, a student who uses structure would identify that the equation is saying 3 times something is 9 and thus the quantity in parenthesis must be 3.

Example problem:

- Solve for x : $\frac{3}{x-1} = \frac{6}{x+3}$

The "typical" approach to the above problem would be to cross multiply and solve; a student who identifies and makes use of structure sees that the left side can be multiplied by 2 to create equivalent numerators… then simply set the denominators equal and solve.

Handout 4.

STANDARDS FOR MATHEMATICAL PRACTICES OBSERVATION TOOL

Overall: The mathematics tasks focus on developing CONCEPTUAL UNDERSTANDING and encouraging ALL students to make sense of the mathematics and to persevere in solving mathematical problems. As you observe, check to see if STUDENTS exhibited the following behaviors in solving mathematics problems and if TEACHERS facilitated these behaviors by providing cognitively demanding tasks and encouraging sense making for ALL students.

Mathematical Practice Standard	Teacher: Actions/Responsibilities	Student: Actions/Responsibilities
1. MAKES SENSE OF PROBLEMS AND PERSEVERES IN SOLVING THEM	**Teacher:** • Provides an open-ended problem with no solution pathway evident and/or non-routine problems with multiple solutions. • Provides time and facilitates discussion in problem solutions. • Facilitates discourse in the classroom so that students UNDERSTAND the approaches of others. • Provides opportunities for students to explain themselves, the meaning of a problem, etc. • Provides opportunities for students to connect concepts to "their" world. • Provides students TIME to think and become "patient" problem solvers. • Facilitates and encourages students to check their answers using different methods (not calculators). • Provides problems that focus on relationships and are "generalizable".	**Students:** • Are actively engaged in solving problems & thinking is visible (i.e., DOING MATHEMATICS vs. FOLLOWING STEPS OR PROCEDURES). • Are analyzing givens, constraints, relationships, and goals (NOT the teacher). • Are discussing with one another, making conjectures, planning a solution pathway, not jumping into a solution attempt or guessing at the direction to take. • Relate current "situation" to concept or skill previously learned and check answers using different methods. • Continually ask self, does this make sense?

STANDARDS FOR MATHEMATICAL PRACTICES OBSERVATION TOOL

Mathematical Practice Standard	Teacher: Actions/Responsibilities	Student: Actions/Responsibilities
2. REASONING ABSTRACTLY AND QUANTITATIVELY	**Teacher:** • Provides a range of representations of math problem situations and encourages various solutions. • Provides opportunities for students to make sense of quantities and their relationships in problem situations. • Provides problems that require flexible use of properties of operations and objects. • Emphasizes quantitative reasoning which entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, **not just how to compute them and/or rules**; and knowing and flexibly using different properties of operations and objects.	**Students:** • Use varied representations and approaches when solving problems. • Make sense of quantities and their relationships in problem situations. • Are *decontextualizing* (abstract a given situation and represent it symbolically and manipulate the representing symbols), and *contextualizing* (pause as needed during the manipulation process in order to probe into the referents for the symbols involved. • Use quantitative reasoning that entails creating a coherent representation of the problem at hand, considering the units involved, and attending to the meaning of quantities, NOT just how to compute them.
3. CONSTRUCTING VIABLE ARGUMENTS AND CRITIQUING THE ARGUMENTS OF OTHERS	**Teacher**: • Uses tasks that allow students to analyze situations by breaking them into cases and then justify, defend/refute and communicate examples and counterexamples, etc. • Provides **ALL** students opportunities to understand and use stated assumptions, definitions, and previously established results in constructing arguments. • Provides ample time for students to make conjectures and build a logical progression of statements to explore the truth of their conjectures. • Provides opportunities for students to construct arguments and critique arguments of peers. • Facilitates and guides students in recognizing and using counterexamples. • Encourages and facilitates students justifying their conclusions, communicating, and responding to the arguments of others. • Asks useful questions to clarify and/or improve students' arguments.	**Students:** • Make conjectures and explore the truth of their conjectures. • Recognize and use counterexamples. • Justify and defend ALL conclusions and communicates them to others. • Recognize and explain flaws in arguments. (After listening or reading arguments of others, they respond by deciding whether or not they make sense. They ask useful questions to improve arguments.) • Elementary Students: construct arguments using concrete referents such as objects, drawings, diagrams, actions. Later, students learn to determine the domains to which an argument applies.

STANDARDS FOR MATHEMATICAL PRACTICES OBSERVATION TOOL

Mathematical Practice Standard	Teacher: Actions/Responsibilities	Student: Actions/Responsibilities
4. MODEL WITH MATHEMATICS	**Teacher:** • Provides problem situations that apply to everyday life. • Provides rich tasks that focus on conceptual understanding, relationships, etc.	**Students:** • Apply the mathematics they know to everyday life, society, and the workplace. • Write equations to describe situations. • Are comfortable in making assumptions and approximations to simplify complicated situations. • Analyze relationships to draw conclusions. • Improve their model if it has not served its purpose.
5. ATTENDS TO PRECISION	**Teacher:** • Facilitates, encourages and expects precision in communication including correct usage of mathematical vocabulary. • Provides opportunities for students to explain and/or write their reasoning to others.	**Students:** • Use and clarify mathematical definitions in discussions and in their own reasoning (orally and in writing). • Use, understand and state the meanings of symbols. • Express numerical answers with a degree of precision.
6. APPROPRIATE TOOLS USED	**Teacher:** • Provides a variety of tools and technology for students to explore to deepen their understanding of math concepts. • Provides problem solving tasks that require students to consider a variety of tools for solving. (Tools might include pencil/paper, concrete models, ruler, protractor, calculator, spreadsheet, computer algebra system, statistical package, or dynamic geometry software, etc.)	**Students:** • Consider available tools when solving a mathematical problem. • Are familiar with a variety of mathematics tools and use them when appropriate to explore and deepen their understanding of concepts.

STANDARDS FOR MATHEMATICAL PRACTICES OBSERVATION TOOL

Mathematical Practice Standard	Teacher: Actions/Responsibilities	Students: Actions/Responsibilities
7. LOOK FOR AND MAKE USE OF STRUCTURE	**Teacher:** • Provides opportunities and time for students to explore patterns and relationships to solve problems. • Provides rich tasks and facilitates pattern seeking and understanding of relationships in numbers rather than following a set of steps and/or procedures.	**Students:** • Look closely to discern patterns or structure. • Associate patterns with properties of operations and their relationships. • Step back for an overview and can shift perspective. • See complicated things, such as algebraic expressions, as single objects or as composed of several objects. (Younger children decompose and compose numbers.)
8. LOOK FOR AND EXPRESS REGULARITY IN REPEATED REASONING	**Teacher:** • Provides problem situations that allow students to explore regularity and repeated reasoning. • Provides rich tasks that encourage students to use repeated reasoning to form generalizations and provides opportunities for students to communicate these generalizations.	**Students:** • Notice if calculations are repeated and look for both general methods and shortcuts. • Pay attention to regularity and use to solve problems. • Use regularity and use this to lead to a general formula and generalizations. • Maintain oversight of the process of solving a problem while attending to details and **continually** evaluates the reasonableness of immediate results.

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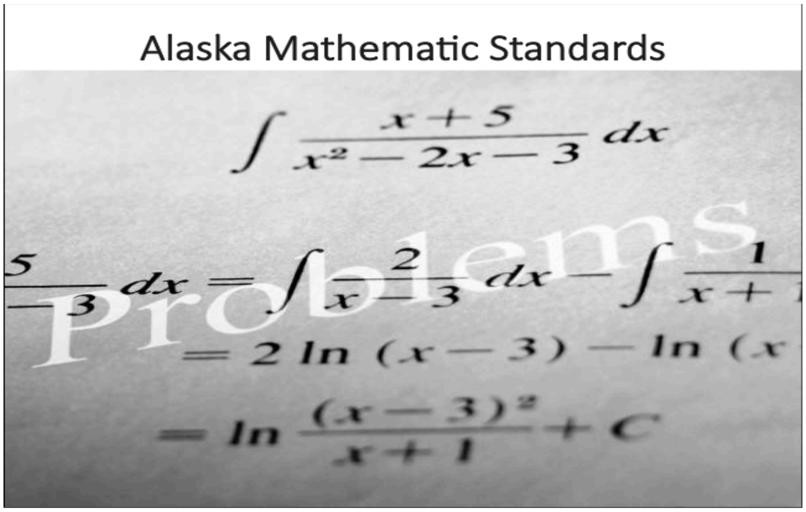


Welcome

Our Goal

Shrink the Change

Brilliantly:

Teach the **focus areas of mathematical content** within your grade band with precision

Integrate standards of mathematical practice into every lesson

Provide **quality performance tasks** within your lessons

General Shifts in Mathematics

- **Focus:** In each grade or course, focus deeply on 2-4 topics
- **Coherence: Think** across grades and link major topics within grades.
- **Rigor:** In major topics pursue with equal intensity
  - Conceptual understanding
  - Procedural skill and Fluency, and
  - Application

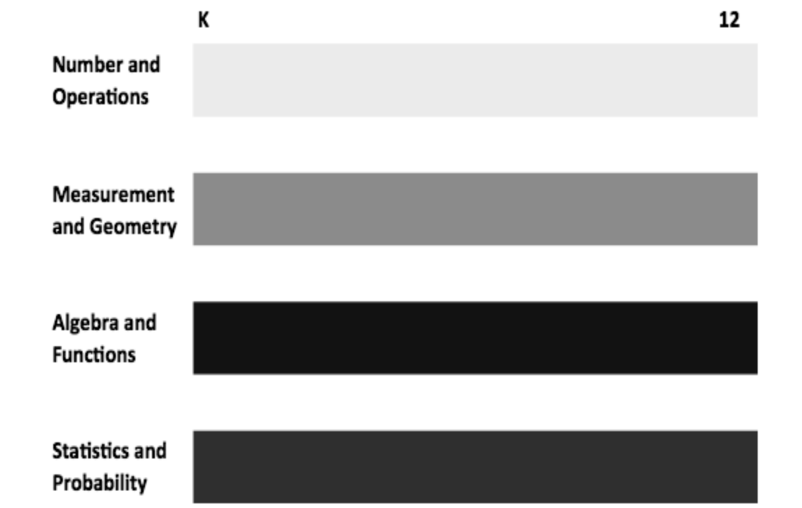
Focus

Why Shift One, Focus

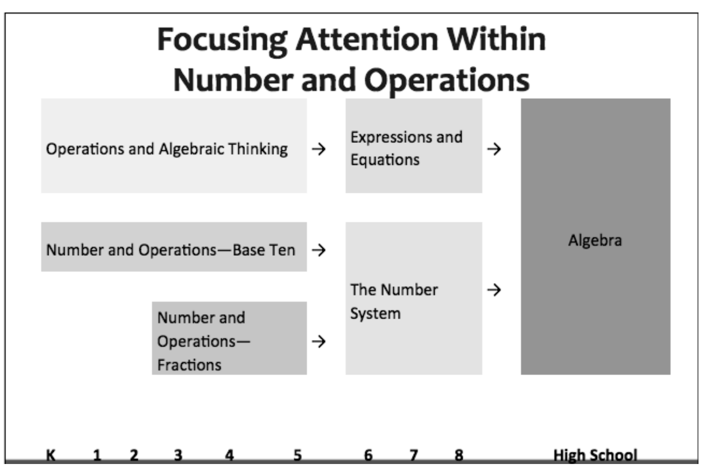
Focus strongly where the standards focus

- Significantly narrow the scope of content and deepen how time and energy is spent in the math classroom
- Focus deeply only on what is emphasized in the standards, so that students gain strong foundations

Traditional U.S. Approach



All Roads Lead to Algebra

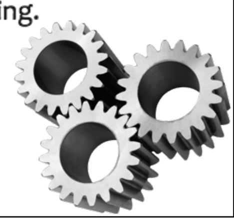


Coherence

Why Shift Two, Coherence

Coherence Think across grades, and link to major topics within grades

- Carefully connect the learning within and across grades so that students can build new understanding onto foundations built in previous years.
- Begin to count on solid conceptual understanding of core content and build on it. Each standard is not a new event, but an extension of previous learning.



Coherence: Think across grades

Fraction example:

"The **coherence** and sequential nature of mathematics dictate the foundational skills that are necessary for the learning of algebra. The most important foundational skill not presently developed appears to be proficiency with fractions (including decimals, percents, and negative fractions).

The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in algebra can be expected."

Final Report of the National Mathematics Advisory Panel (2008, p. 18)

Rigor

Why Shift Three, Rigor

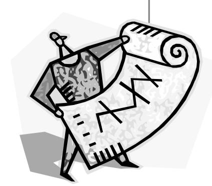
Rigor In major topics, pursue conceptual understanding, procedural skill and fluency, and application

- The Alaska Mathematics Standards require a balance of:
  - Solid conceptual understanding
  - Procedural skill and fluency
  - Application of skills in problem solving situations
- This requires equal intensity in time, activities, and resources in pursuit of all three

Handout 1

Solid Conceptual Understanding

- Teach more than “how to get the answer” and instead support students’ ability to access concepts from a number of perspectives
- Students are able to see math as more than a set of mnemonics or discrete procedures
- Conceptual understanding supports the other aspects of rigor (fluency and application)



Priorities in Mathematics

Grade	Priorities in Support of Rich Instruction and Expectations of Fluency and Conceptual Understanding
K–2	Addition and subtraction, measurement using whole number quantities
3–5	Multiplication and division of whole numbers and fractions
6	Ratios and proportional reasoning; early expressions and equations
7	Ratios and proportional reasoning; arithmetic of rational numbers
8	Linear algebra

Focus and Priorities of Rigor

Alaska Mathematics Standards | Shift #3: Rigor – An Equation

Rigor = *Conceptual understanding + Procedural skill and fluency + Application*

Shift #3: Rigor requires a balance of the three discrete components of math instruction: conceptual understanding, procedural skills and fluency, and application. This is not simply a pedagogical option, but is required by the Standards. The majority of the Standards specifically call for conceptual understanding, fluency, or application, but not every standard will necessarily fit neatly into just one of these three discrete components. For example, certain standards can be said to require procedural skill and conceptual understanding.

Grade	Standard	Procedural skill and Conceptual Understanding Standards Examples
3	3.G.2	Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape.*
6	6.EE.1	Write and evaluate numerical expressions involving whole-number exponents. *For example, multiply by powers of 10 and products of numbers using exponents ($7•7•7 = 7^3$).*

Conceptual Understanding

Conceptual understanding: The Standards call for conceptual understanding of key concepts, such as place value and ratios. Teachers support students' ability to access concepts from a number of perspectives so that students are able to see math as more than a set of mnemonics or discrete procedures. Conceptual understanding standards often use the terms "understand" and "recognize."

Grade	Standard	Deep Conceptual Understanding Standards Examples
3	3.NBT.1	Use place value understanding to round whole numbers to the nearest 10 or 100.
6	6.NS.5	Understand that positive and negative numbers describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explain the meaning of 0 in each situation.

Procedural Skill and Fluency

Alaska Mathematics Standards — Shift #3: Rigor – An Equation

Procedural skill and fluency: The Standards call for speed and accuracy in calculation. Teachers structure class time and/or homework time for students to practice core functions such as single-digit multiplication so that students have access to more complex concepts and procedures. Fluency standards clearly state "fluently" in the standard.

Required Fluencies K-6

Grade	Standard	Required Fluency
K	K.OA.5	Add/subtract up to 5
1	1.OA.6	Add/subtract up to 10
2	2.OA.2 2.NBT.5	Add/subtract up to 20 (know single-digit sums from memory) Add/subtract up to 100
3	3.OA.7 3.NBT.2	Multiply/divide up to 100 (know single-digit products from memory) Add/subtract up to 1000
4	4.NBT.4	Add/subtract up to 1,000,000
5	5.NBT.5	Multi-digit multiplication
6	6.NS.2,3	Multi-digit division Multi-digit decimal operations

Application

Application: The Standards call for students to use math flexibly for applications. Teachers provide opportunities for students to apply math in context. Teachers in content areas outside of math, particularly science, ensure that students are using math to access and make meaning of content. Application standards typically state "apply" or "solve."

Grade	Standard	Application Grade 3 and 6 Standards Examples
3	3.MD.1	Tell and write time to the nearest minute and measure time intervals in minutes. Solve word problems involving addition and subtraction of time intervals in minutes or hours (e.g., by representing the problem on a number line diagram or clock).
6	6.SP.4	Display numerical data in plots on a number line, including dot or line plots, histograms, and box (box and whisker) plots.

What Does Rigor Look Like?

- **Teachers plan for what students should understand and be able to do by the end of the learning cycle.**
- **Scaffold instruction from facts and details to robust generalizations and processes.**
- **Provide a clear progression of learning, more opportunities to apply their knowledge and make inferences based on what they are learning.**
- **The shift requires students to make and defend claims with sound evidence including grounds, backing, and qualifiers as part of utilizing the knowledge they acquire in class.**

Instructional Strategies to Achieve Rigor

- Identifying Critical Content
- Previewing New Content
- Organizing Students to Interact with content
- Helping Students Process Content
- Helping Student Elaborate on Content
- Helping Students Record and Represent Knowledge
- Managing Response Rates with Tiered Questioning Techniques
- Reviewing Content
- Helping Students Practice Skills, Strategies, and Processes
- Helping Students Examine Similarities and Differences
- Helping Students Examine Their Reasoning
- Helping Students Revise Knowledge
- Helping Students Engage in Cognitively Complex Tasks

Modes of Engagement

Core Instructional Actions

- **Core Action 1**:
  - **Aligned Lessons:** Ensure the work of the lesson reflects the shifts in the standards.
- **Core Action 2:**
  - **Instructional Delivery**: Employ instructional practices that allow all students to master the content of the lesson.
- **Core Action 3**:
  - **Practice Content-Connection**: Provide all students with opportunities to exhibit mathematical practices in connection with the content of the lesson.

Core Action 1: Aligned Lessons

- **Core Action 1**: Insure the work of the lesson reflects the shifts required in the standards for mathematics.

A. The lesson focuses on grade-level cluster(s), grade level content standard(s) or part(s) thereof.
B. The lesson reflects the full intent of the grade-level cluster(s), grade-level content standard(s) or part(s) therof being addressed
C. The lesson intentionally relates new concepts to students' prior skills and knowledge.
D. The lesson intentionally targets the aspect(s) of rigor (conceptual understanding, procedural skill and fluency, application) called for by the standard(s) being addressed.

Key Components of Core Action 1:

- Lessons are designed to include the Three Shifts of:
  - **Focus**
  - **Coherence**
  - **Rigor**
    - Conceptual Understanding
    - Procedural Skill and Fluency
    - Application

Example of a Lesson timeline

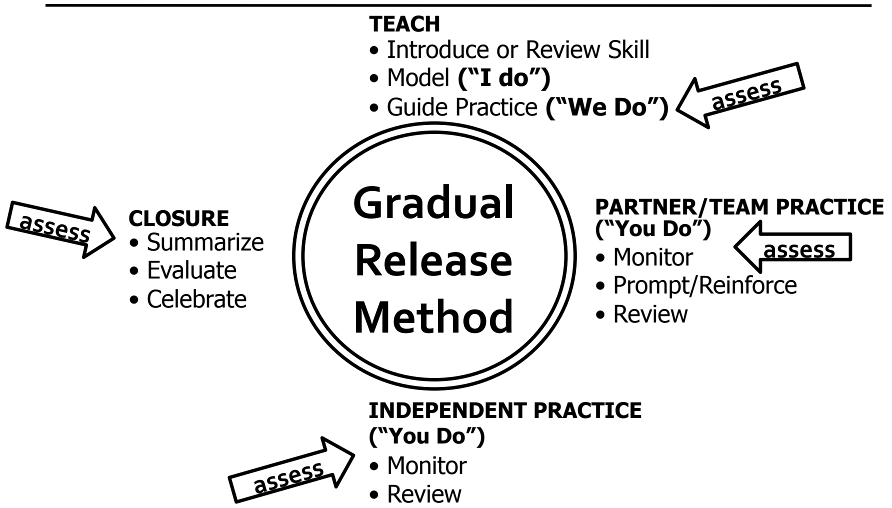
Lesson 3

Objective: Add fractions with unlike units using the strategy of creating equivalent fractions.

Suggested Lesson Structure

Fluency Practice	(12 minutes)
Application Problem	(5 minutes)
Concept Development	(33 minutes)
Student Debrief	(10 minutes)
Total Time	**(60 minutes)**

What Does Core Action 1 Look Like?



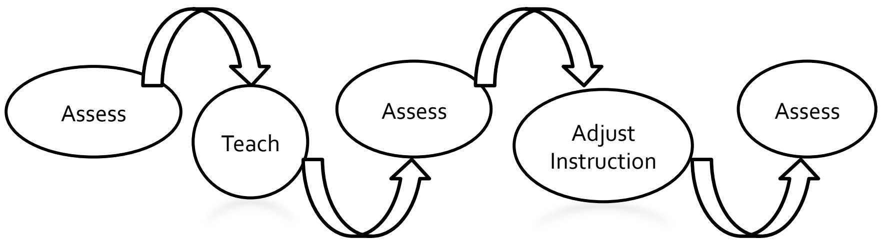
Core Action 2: Instructional Delivery

Core Action 2: Employ instructional practices that allow all students to master the content of the lesson.

A. The teacher uses explanations, representations, and/or examples to make the mathematics of the lesson explicit.
B. The teacher poses high quality questions and problems that prompts students to share their developing thinking about the content of the lesson.
C. The teacher provides time for students to work with and practice grade-level problems and exercises.
D. The teacher uses variations in students' solution methods to strengthen other students' understanding of the content.
E. The teacher checks for understanding throughout the lesson, using informal, but deliberate methods (such as questioning or assigning short problems)
F. The teacher guides student thinking toward the focus of the lesson and summarizes the mathematics with reference to student work and discussion.

Plan for clarifying misunderstanding

- https://www.teachingchannel.org/videos/class-warm-up-routine



Core Action 3: Practice-Content Connections

- **Core Action 3**: Provide all students with opportunities to exhibit mathematical practices in connection with the content of the lesson.
  A. The teacher uses strategies to keep all students persevering with challenging problems.
  B. The teacher establishes a classroom culture in which students explain their thinking.
  C. The teacher orchestrates conversations in which students talk about each other's thinking.
  D. The teacher connects students' informal language to precise mathematical language appropriate.
  E. The teacher has established a classroom culture in which students choose and use appropriate tools when solving a problem.
  F. The teacher asks students to explain and justify work and provides feedback that helps students revise initial work.

Let's Take a Look…

- https://www.teachingchannel.org/videos/multiplying-fractions-lesson

- Turn to your elbow partner and discuss the following:
  - Why does Ms. Pittard present students with a variety of solutions
  - What did you learn from Ms. Pittard about engaging all students?

Mathematical Practice Standards

Grouping of Mathematical Practices

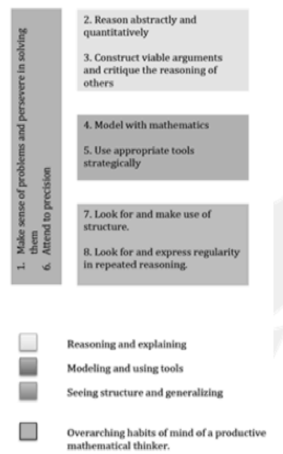
Reasoning and Explaining
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others

Modeling and Using Tools
4. Model with mathematics
5. Use appropriate tools strategically

Seeing Structure and Generalizing
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Overarching Habits of Mind of a Productive Mathematical Thinker
1. Make sense of problems and persevere in solving them
6. Attend to precision

Adapted from (McCallum, 2011)



Handout 2

Implementation Characteristics

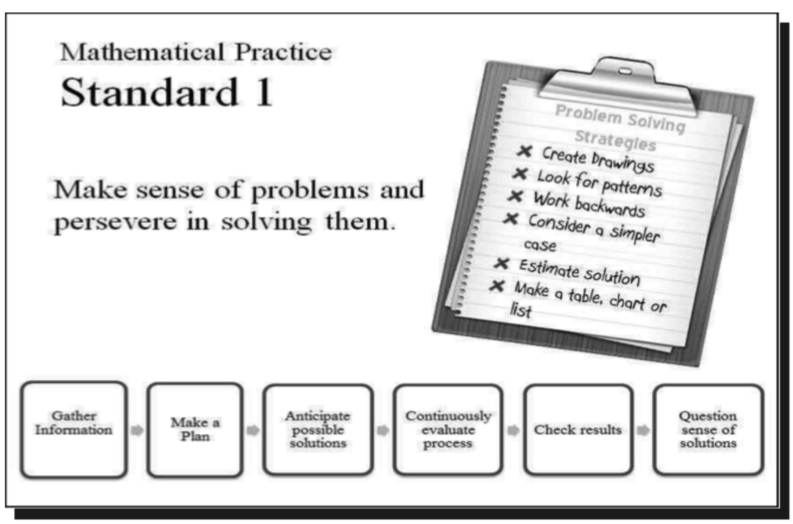
- What Does it look like in planning and delivery
  - Task: Elements to keep in mind when determining learning experiences
    - Ex. Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding
  - Teacher: Actions that further the development of math practices within students
    - Ex. Allows students time to initiate a plan; uses questions to prompt students as need to assist students in developing a pathway

Handout 3

1. Make sense of problems and persevere in solving them.

- Explain the meaning of a problem
- Look for entry to its solution
- Plan a solution pathway
- Monitor and evaluate their progress
- Make course corrections if needed
- Continually ask themselves, "Does this make sense?"

Attributes of Mathematical Practice 1.



Let's Take a Look…

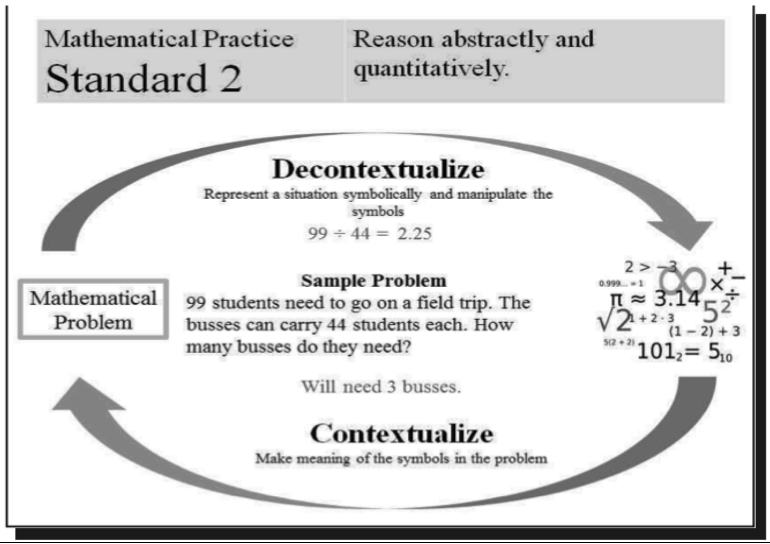
- https://www.teachingchannel.org/videos/math-practice-standard-perseverance
- How does Ms. Noonan differentiate to present appropriate challenges to each of her students?
- What kinds of feedback does Ms. Noonan give her students while they work?
- What is the effect of explicitly focusing on perseverance skills?

2. Reason abstractly and quantitatively.

- Make sense of quantities and their relationship within the problem
- Ability to de-contextualize
  - To abstract a given situation and represent it symbolically

- Ability to contextualize
  - Pause as needed during the manipulation process in order to probe into the referents for the symbols involved
- Habits of creating a coherent representation of the problem at hand

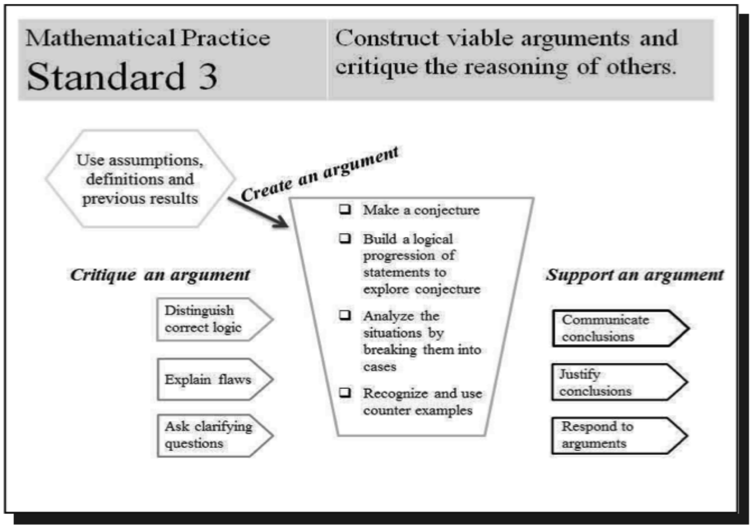
Attributes of Mathematical Practice 2.



3. Construct viable arguments and critique the reasoning of others.

- Understand and use stated assumptions, definitions, and previously established results in constructing arguments.
- Justify and communicate conclusions to others
- Reason inductively about data
- Construct arguments using concrete referents

Attributes of Mathematical Practice 3.



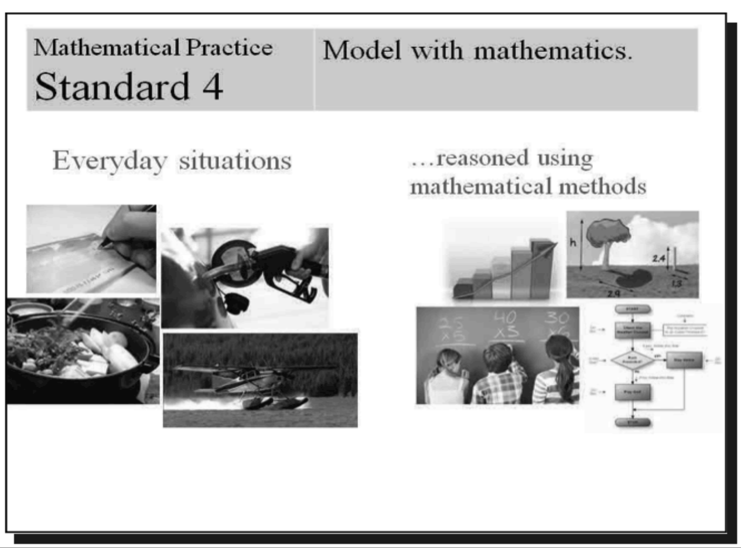
Let's Take a Look…

- https://www.teachingchannel.org/videos/conjecture-lesson-plan
- Turn to your elbow partner and share what you learned from Ms. McPhillips

4. Model With Mathematics

- Apply mathematics to solve problems in every day life
- Make assumptions and approximations to simplify a complicated situation
- Identify important quantities in a practical situation
- Map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.
- Routinely interpret mathematical results in their context of the situation

Attributes of Mathematical Practice 4.



Let's take a look…

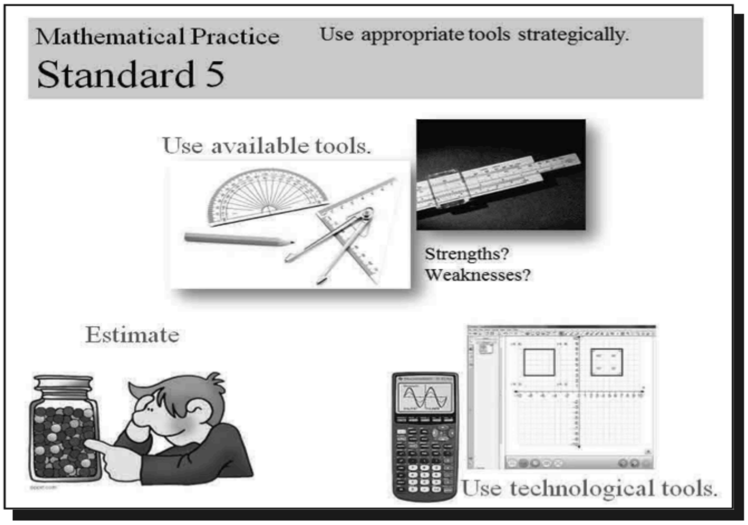
https://www.teachingchannel.org/videos/real-world-geometry-lesson

- Why does Ms. Park go back to 3rd grade content standards? How does she ramp up to teaching 6th grade content?
- When Ms. Park says that you create your practice by knowing the content first, what does this mean?
- How does Ms. Park ask students to reflect on their learning? Why is this reflection important?

5. Use appropriate tools strategically

- Consider the available tools when solving a mathematical problem
- Familiar with tools appropriate for their grade or course to make sound decisions
- Detect possible errors by using mathematical knowledge
- Visualize the results of varying assumptions, explore consequences, and compare predictions with data

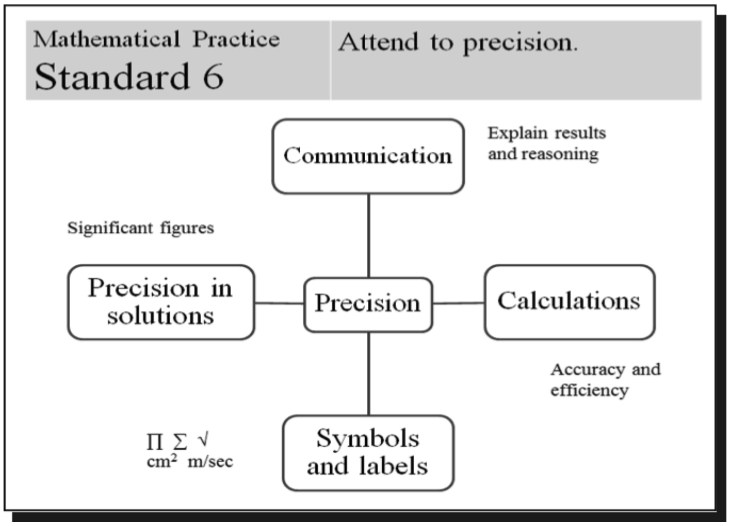
Attributes of Mathematical Practice 5.



6. Attend to precision.

- Use clear definitions in discussion with others
- learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element
- Carefully specify units of measure
- Calculate accurately and efficiently

Attributes of Mathematical Practice 6.



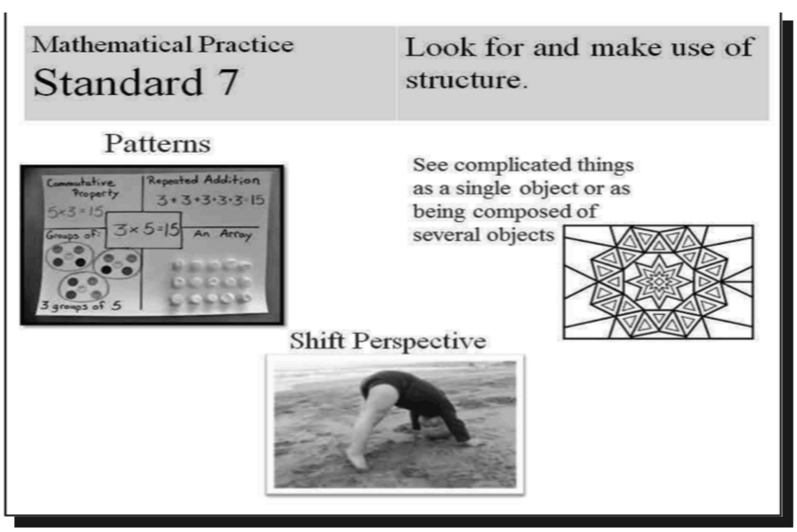
Let's Take a Look…

- https://www.teachingchannel.org/videos/student-self-correction
- How does this strategy encourage independence?
- What kinds of questions do Ms. Brookins and Mr. James ask their students?
- How could this strategy be used to help students critique the work of others?

7. Look for and make use of structure.

- Look closely to discern a pattern or structure
- Step back, look from a perspective of an overview and make adjustments
- See complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.

Attributes of Mathematical Practice 7.



Let's Take a Look…

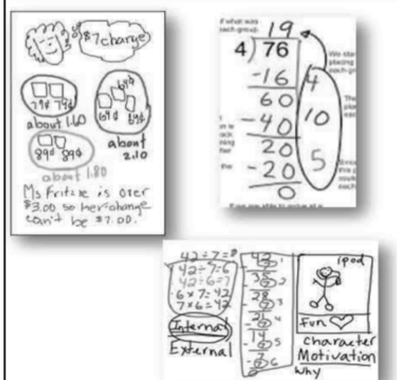
- www.teachingchannel.org/videos/sorting-classifying-equations-overview
- What structures does Ms. Warburton create to facilitate strong collaboration?
- How does Ms. Warburton encourage deep understanding through questioning, encouragement, and reflection?
- What does Ms. Warburton mean when she tells students to "ask the math?"

8. Look for and express regularity regularity in repeated reasoning.

- Notice if calculations are repeated
- Maintain oversight of the process, while attending to the details.
- Evaluate the reasonableness of their intermediate results.

Attributes of Mathematical Practice 8.

Mathematical Practice Standard 8	Look for and express regularity in repeated reasoning.



- See repeated calculations and look for generalizations
- Recognize reasonable solutions
- See the process – attend to details
- Understand the broader application of patterns

Observing the Mathematical Practices

Implementation Characteristics

Implementing Standards for Mathematical Practices

#1 Make sense of problems and persevere in solving them.

Summary of Standards for Mathematical Practice	Questions to Develop Mathematical Thinking
1. Make sense of problems and persevere in solving them. • Interpret and make meaning of the problem looking for starting points. Analyze what is given to explain to themselves the meaning of the problem. • Plan a solution pathway instead of jumping to a solution. • Monitor the progress and change the approach if necessary. • See relationships between various representations. • Relate current situations to concepts or skills previously learned and connect mathematical ideas to one another. • Students ask themselves, "Does this make sense?" and understand various approaches to solutions.	How would you describe the problem in your own words? How would you describe what you are trying to find? What do you notice about...? What information is given in the problem? Describe the relationship between the quantities. Describe what you have already tried. What might you change? Talk me through the steps you've used to this point. What steps in the process are you most confident about? What are some other strategies you might try? What are some other problems that are similar to this one? How might you use one of your previous problems to help you begin? How else might you organize...represent...show...?

Implementation Characteristics: What does it look like in planning and delivery?

Task: elements to keep in mind when determining learning experiences **Teacher:** actions that further the development of math practices within their students

Task:

- ☐ Requires students to engage with conceptual ideas that underlie the procedures to complete the task and develop understanding.
- ☐ Requires cognitive effort - while procedures may be followed, the approach or pathway is not explicitly suggested by the task, or task instructions and multiple entry points are available. The problem focuses students' attention on a mathematical idea and provides an opportunity to develop and/or use mathematical habits of mind.
- ☐ Allows for multiple entry points and solution paths as well as, multiple representations, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations to develop meaning.
- ☐ Requires students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- ☐ Requires students to defend and justify their solutions.

Teacher:

- ☐ Allows students time to initiate a plan; uses question prompts as needed to assist students in developing a pathway.
- ☐ Continually asks students if their plans and solutions make sense.
- ☐ Questions students to see connections to previous solution attempts and/or tasks to make sense of current problem.
- ☐ Consistently asks to defend and justify their solution by comparing solution paths.
- ☐ Differentiates to keep advanced students challenged during work time

Institute for Advanced Study/Park City Mathematics Institute/ Created by Learning Services, Modified by Melisa Hancock, 2013

Refer to Handout 3.

Looks Like – Sounds Like

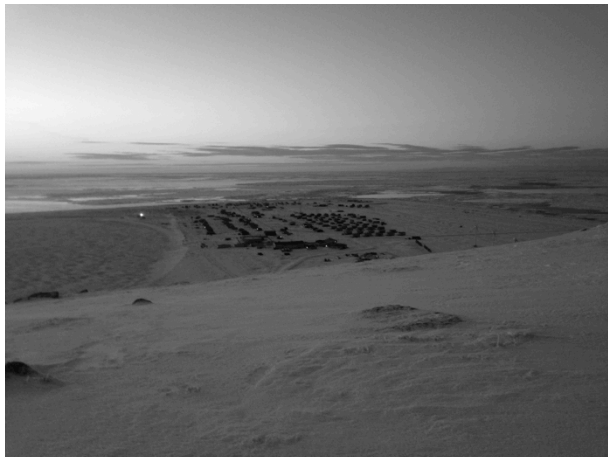
STANDARDS FOR MATHEMATICAL PRACTICES OBSERVATION TOOL

Overall: The mathematics tasks focus on developing CONCEPTUAL UNDERSTANDING and encouraging ALL students to make sense of the mathematics and to persevere in solving mathematical problems. As you observe, check to see if STUDENTS exhibited the following behaviors in solving mathematics problems and if TEACHERS facilitated these behaviors by providing cognitively demanding tasks and encouraging sense making for ALL students.

Mathematical Practice Standard	Teacher: Actions/Responsibilities	Student: Actions/Responsibilities
1. MAKES SENSE OF PROBLEMS AND PERSEVERES IN SOLVING THEM	**Teacher:** • Provides an open-ended problem with no solution pathway evident and/or non-routine problems with multiple solutions. • Provides time and facilitates discussion in problem solutions. • Facilitates discourse in the classroom so that students UNDERSTAND the approaches of others. • Provides opportunities for students to explain themselves, the meaning of a problem, etc. • Provides opportunities for students to connect concepts to "their" world. • Provides students TIME to think and become "patient" problem solvers. • Facilitates and encourages students to check their answers using different methods (not calculators). • Provides problems that focus on relationships and are "generalizable".	**Students:** • Are actively engaged in solving problems & thinking is visible (i.e., DOING MATHEMATICS vs. FOLLOWING STEPS OR PROCEDURES). • Are analyzing givens, constraints, relationships, and goals (NOT the teacher). • Are discussing with one another, making conjectures, planning a solution pathway, not jumping into a solution attempt or guessing at the direction to take. • Relate current "situation" to concept or skill previously learned and check answers using different methods. • Continually ask self, does this make sense?

Handout 4

Wrap Up



Standards for Mathematical Practice: Commentary and Elaborations for K–5

12 February 2014

Suggested citation:

Illustrative Mathematics. (2014, February 12). *Standards for Mathematical Practice: Commentary and Elaborations for K–5*. Tucson, AZ.

For discussion of the Elaborations and related topics, see the Tools for the Common Core blog: http://commoncoretools.me.

The Standards for Mathematical Practice, annotated for the K–5 classroom

The Common Core State Standards describe the Standards for Mathematical Practice this way:

> The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report Adding It Up: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

In this document we provide two different ways of adapting the language of the practice standards to the K–5 setting.

In this section we provide annotated versions of the standards that provide additional interpretation of the standards appropriate for K–5 classrooms. This section is intended for people who want to understand how the original language of the standards applies in K–5.

In the next section we provide *elaborations* of the standards: narrative descriptions that integrate the annotations from the first section and provide a coherent description of how the practice standards play out in the K–5 classroom.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution.• They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution.• They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem.• Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?"• They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

• Young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started.

• For example, to solve a problem involving multidigit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.

• Mathematically proficient elementary students may consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches.

• When they find that their solution pathway does not make sense, they look for another pathway that does.

2. Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize•—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand;• considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

• For example, to find the area of the floor of a rectangular room that measures 10 m. by 12 m., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as $10 \times 12 = 120$. He has decontextualized the problem. When he states at the end that the area of the room is 120 square meters, he has contextualized the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (MP.4).

• For example, when a student sees the expression $40 - 26$, she might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, she thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. She then uses what she did in the context to identify the solution of the original abstract problem.

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments.• They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples.• They justify their conclusions, communicate them to others, and respond to the arguments of others.• They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions.• Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies.• Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

• For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle.

• For example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

• Students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning).

• For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older elementary student might use an area representation to show why the distributive property holds.

• For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands.

In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

4. Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace.• In early grades, this might be as simple as writing an addition equation to describe a situation.• In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas.• They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.•

• For elementary students, this includes the contextual situations they encounter in the classroom. When elementary students are first studying an operation such as addition, they might arrange counters to solve problems such as this one: there are seven animals in the yard, some are dogs and some are cats, how many of each could there be? They are using the counters to model the mathematical elements of the contextual problem—that they can split a set of 7 into a set of 3 and a set of 4. When they learn how to write their actions with the counters in an equation, $4 + 3 = 7$, they are modeling the situation with numbers and symbols. Similarly, when students encounter situations such as sharing a pan of cornbread among 6 people, they might first show how to divide the cornbread into 6 equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1/6, they are now modeling the situation with mathematical notation.

• In addition to numbers and symbols, elementary students might use geometric figures, pictures or physical objects used to abstract the mathematical elements of the situation, or a mathematical diagram such as a number line, a table, or a graph, or students might use more than one of these to help them interpret the situation.

• For example, if there is a Penny Jar that starts with 3 pennies in the jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, $4 \times 10 + 3$.

• As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

5. Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software.• Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations.• For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.•

• The tools that elementary students might use include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.), drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.), paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives or other available technologies.

• Mathematically proficient elementary students choose tools that are relevant and useful to the problem at hand. These include such tools as are mentioned above as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve $3/5 - \frac{1}{2}$, a student might recognize that knowledge of equivalents of $\frac{1}{2}$ is an appropriate tool: since $\frac{1}{2}$ is equivalent to $2\frac{1}{2}$ fifths, the result is $\frac{1}{2}$ of a fifth or $\frac{1}{10}$.

• This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

6. Attend to precision.

Mathematically proficient students try to communicate precisely to others.• They try to use clear definitions in discussion with others and in their own reasoning.When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient elementary students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.• They are careful about specifying units of measure,• and labeling axes to clarify the correspondence with quantities in a problem.• They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

• Elementary students start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, “it works" without explaining what “it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it.

• For example, the equivalence of 8 and $5+3$ can be written both as $5 + 3 = 8$ and $8 = 5 + 3$. Similarly, the equivalence of $6 + 2$ and $5 + 3$ is expressed as $6 + 2 = 5 + 3$.

• When measuring, mathematically proficient elementary students use tools and strategies to minimize the introduction of error. From Kindergarten on, they count accurately, using strategies so that they include each object once and only once without losing track. They calculate accurately and efficiently and use clear and concise notation to record their work.

• In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.

7. Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure.• Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

• Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older elementary students calculate 16×9, they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10 + 6) \times 9 = (10 \times 9) + (6 \times 9)$. To determine the volume of a $3 \times 4 \times 5$ rectangular prism, students might see the structure of the prism as five layers of 3×4 arrays of cubes.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts.• Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal.• By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y-2)/(x-1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x-1)(x+1)$, $(x-1)(x^2+x+1)$, and $(x-1)(x^3+x^2+x+1)$ might lead them to the general formula for the sum of a geometric series.• As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

• For example, younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call "opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow.

• Students in the middle elementary grades might notice a pattern in the change to the product when a factor is increased by 1: $5 \times 7 = 35$ and $5 \times 8 = 40$—the product changes by 5; $9 \times 4 = 36$ and $10 \times 4 = 40$—the product changes by 4. Students might then express this regularity by saying something like, "When you change one factor by 1, the product increases by the other factor."

• Mathematically proficient elementary students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor; or that, whenever they toss counters, for each combination that comes up, its "opposite" can also come up. As students practice articulating their observations, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).

K–5 elaborations of the standards for mathematical practice

The following elaborations of the practice standards integrate the commentary from the previous section into a single narrative describing a K–5 version of each standard for mathematical practice. This way of looking at the standards might be more useful than the previous secion in working with K–5 teachers.

1. Make sense of problems and persevere in solving them.

Mathematically proficient students at the elementary grades explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. As they work, they continually ask themselves, "Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, to solve a problem involving multidigit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach.

Mathematically proficient students consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches. They can explain correspondences among physical models, pictures or diagrams, equations, verbal descriptions, tables, and graphs.

2. Reason abstractly and quantitatively.

Mathematically proficient students at the elementary grades make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects.

Mathematically proficient students can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, when a student sees the expression $40-26$, she might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, she thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. She then uses what she did in the context to identify the solution of the original abstract problem.

Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context. For example, to find the area of the floor of a rectangular room that measures 10 m. by 12 m., a student might represent the problem as an equation, solve it mentally, and record the problem and solution as $10 \times 12 = 120$. He has decontextualized the problem. When he states at the end that the area of the room is 120 square meters, he has contextualized the answer in order to solve the original problem. Problems like this that begin with a context and are then represented with mathematical objects or symbols are also examples of modeling with mathematics (MP.4).

3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient students at the elementary grades construct mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. For example, in order to demonstrate what happens to the sum when the same amount is added to one addend and subtracted from another, students in the early grades might represent a story about children moving between two classrooms: the number of children in each classroom is an addend; the total number of children in the two classrooms is the sum. When some students move from one classroom to the other, the number of students in each classroom changes by that amount—one addend decreases by some amount and the other addend increases by that same amount—but the total number of students does not change. An older student might use an area representation to show why the distributive property holds. Arguments may also rely on definitions, previously established results, properties, or structures. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor.

Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8, Look for and express regularity in repeated reasoning). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. For example, students might make an argument based on an area representation of multiplication to show that the distributive property applies to problems involving fractions.

Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments, compare them to others, and reconsider their own arguments in response to the critiques of others.

4. Model with mathematics.

When given a problem in a contextual situation, mathematically proficient students at the elementary grades can identify the mathematical elements of a situation and create a mathematical model that shows those mathematical elements and relationships among them. The mathematical model might be represented in one or more of the following ways: numbers and symbols, geometric figures, pictures or physical objects used to abstract the mathematical elements of the situation, or a mathematical diagram such as a number line, a table, or a graph, or students might use more than one of these to help them interpret the situation.

For example, when students are first studying an operation such as addition, they might arrange counters to solve problems such as this one: there are seven animals in the yard, some are dogs and some are cats, how many of each could there be? They are using the counters to model the mathematical elements of the contextual problem—that they can split a set of 7 into a set of 3 and a set of 4. When they learn how to write their actions with the counters in an equation, $4 + 3 = 7$, they are modeling the situation with numbers and symbols. Similarly, when students encounter situations such as sharing a pan of cornbread among 6 people, they might first show how to divide the cornbread into 6 equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1/6, they are now modeling the situation with mathematical notation.

Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multistep problems or problems involving more than one variable. For example, if there is a Penny Jar that starts with 3 pennies in the jar, and 4 pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, $4 \times 10 + 3$.

Mathematically proficient students use their model to analyze the relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

5. Use appropriate tools strategically.

Mathematically proficient students at the elementary grades consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.), drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.), paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives or other available technologies. Proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations.

Mathematically proficient students choose tools that are relevant and useful to the problem at hand. These include such tools as are mentioned above as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve $3/5 - \frac{1}{2}$, a student might recognize that knowledge of equivalents of $\frac{1}{2}$ is an appropriate tool: since $\frac{1}{2}$ is equivalent to $2\frac{1}{2}$ fifths, the result is $\frac{1}{2}$ of a fifth or $\frac{1}{10}$.

This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

6. Attend to precision.

Mathematically proficient students at the elementary grades communicate precisely to others. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, "it works" without explaining what "it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using representations, such as pictures, tables, graphs, or diagrams, they use appropriate labels to communicate the meaning of their representation.

When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations.

Elementary students learn to use mathematical symbols correctly and can describe the meaning of the symbols they use. In particular, they understand that the equal sign denotes that two quantities have the same value, and can use it flexibly to express equivalences. For example, the equivalence of 8 and $5+3$ can be written both as $5+3=8$ and $8=5+3$. Similarly, the equivalence of $6+2$ and $5+3$ is expressed as $6+2=5+3$.

When measuring, mathematically proficient students use tools and strategies to minimize the introduction of error. From Kindergarten on, they count accurately, using strategies so that they include each object once and only once without losing track. Mathematically proficient students calculate accurately and efficiently and use clear and concise notation to record their work.

7. Look for and make use of structure.

Mathematically proficient students at the elementary grades use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, the less you subtract, the greater the difference), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16×9, they might apply the structure of place value and the distributive property to find the product: $16 \times 9 = (10+6) \times 9 = (10 \times 9) + (6 \times 9)$. To determine the volume of a $3 \times 4 \times 5$ rectangular prism, students might see the structure of the prism as five layers of 3×4 arrays of cubes.

8. Look for and express regularity in repeated reasoning.

Mathematically proficient students at the elementary grades look for regularities as they solve multiple related problems, then identify and describe these regularities. For example, students might notice a pattern in the change to the product when a factor is increased by 1: $5 \times 7 = 35$ and $5 \times 8 = 40$—the product changes by 5; $9 \times 4 = 36$ and $10 \times 4 = 40$—the product changes by 4. Students might then express this regularity by saying something like, "When you change one factor by 1, the product increases by the other factor." Younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call "opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow. Mathematically proficient students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor; or that, whenever they toss counters, for each combination that comes up, its "opposite" can also come up. As students practice articulating their observations, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).